JOURNAL OF APPROXIMATION THEORY 71, 252–262 (1992)

Almost Everywhere Summability of Orthogonal Polynomial Expansions on the Unit Circle*

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Communicated by Paul Nevai

Received September 30, 1991; accepted October 3, 1991

A condition is obtained for orthogonal polynomials on the unit circle to be almost everywhere strongly $(C, 1)$-summable. © 1992 Academic Press, Inc.

1. INTRODUCTION AND THE MAIN RESULT

Given a finite positive Borel measure μ on the interval $[-\pi, \pi)$ with an infinite set as its support, one defines the polynomials $\phi_n(\zeta) = \phi_n(\mu, \zeta) = \kappa_n \zeta^n + \cdots$ orthonormal on the unit circle with respect to μ by requiring that $\kappa_n = \kappa_n(\mu) > 0$ and the relations

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_m(e^{i\theta}) \overline{\phi_n(e^{i\theta})}\, d\mu(\theta) = \delta_{mn} \tag{1}$$

hold for all $m, n \geqslant 0$, where $\delta_{mn} = 1$ if $m = n$, and $\delta_{mn} = 0$ otherwise, and the bar indicates complex conjugation. The aim of these notes is to prove the following:

THEOREM 1. *Let μ be a finite positive Borel measure on the interval $[-\pi, \pi)$ with infinite support, and let $\phi_n = \phi_n(\mu)$ be the corresponding orthonormal polynomials on the unit circle. Assume*

$$\sum_{\nu=n}^{\infty} |\phi_\nu(0)|^2 = O\left(\frac{1}{n}\right) \tag{2}$$

* This material is based upon work supported by the National Science Foundation under Grant No. DMS-9001154 and by the PSC-CUNY Research Award Program of the City University of New York under Grant No. 662344. The work was performed while the author was visiting The Institute for Advanced Study in Princeton, New Jersey, during the summer of 1991.

0021-9045/92 $5.00

as $n \to \infty$. Assume, further, that f belongs to the closed subspace of L^2_μ spanned by polynomials of $e^{i\theta}$. Then the orthogonal expansion of $f(\theta)$ with respect to the polynomials $\phi_n(e^{i\theta})$ is almost everywhere strongly $(C, 1)$-summable to $f(\theta)$. That is, if

$$f(\theta) \sim \sum_{j=0}^{\infty} c_j \phi_j(e^{i\theta})$$

is the orthogonal expansion of f and $s_\nu(\theta)$ are the partial sums of this series, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\nu=0}^{n-1} |s_\nu(t) - f(t)| = 0 \tag{3}$$

holds for almost every $t \in [-\pi, \pi)$.

Almost every and almost everywhere here and below are meant with respect to the Lebesgue measure, unless otherwise mentioned. In [2, Satz I, p. 84] (restated as [3, Theorem IV.3.1, p. 148]), G. Freud, improving a similar result of K. Tandori [9, Sätze 1 and 2, p. 74], obtained a remarkable result connecting the behavior of the Christoffel function with the strong summability of orthogonal polynomial expansions. His result concerns polynomials orthogonal on an interval of the real line. G. Alexits [1, Theorem 3.4.1, p. 206] extends this result to polynomial-like orthogonal systems (see [1, Sect. 3.1, p. 177]), but polynomial-like systems only generalize real orthogonal polynomials (and some other systems, such as the trigonometric system), not orthogonal polynomials on the unit circle. An important question left open by Freud's result was under what circumstances the assumptions in his theorem are satisfied. An advance in this direction was made in [6, Theorem 2, p. 147], where it was shown that Freud's condition was fulfilled almost everywhere for polynomials in the Szegő class (more precise results are given in [7, Theorems 1, 5, 7, and 8]).

Orthogonal polynomials on the unit circle are often better behaved than the corresponding orthogonal polynomials on the interval $[-1, 1]$; the above result seems to be an exception. Namely, the quoted result in [2] combined with the result mentioned in [6] establishes almost everywhere strong $(C, 1)$-summability of L^2 orthogonal polynomial expansions provided these polynomials belong to the Szegő class of the interval $[-1, 1]$, whereas condition (2) requires much more than that the polynomials belong to the Szegő class of the unit circle. Condition (2) is connected with the behavior of the integral modulus of continuity of the Szegő function of μ; see [4, especially Formula (2.8) on p. 21 and Formulas (3.8) and (3.9) on p. 32; 5, Formula (XII.5), p. 95] (the latter refers to a paper of Freud; we were unable to see how the formula in question is substantiated by Freud's quoted paper).

In the proof of this theorem we establish (3) for every value of $t \in (-\pi, \pi)$ for which

$$F_t(h) = \int_t^{t+h} |f(\theta) - f(t)|^2 \, d\mu(\theta) = o(|h|) \qquad (h \to 0) \tag{4}$$

and

$$\sup_n \frac{1}{n} \sum_{\nu=0}^{n-1} |\phi_\nu(e^{it})|^2 < \infty \tag{5}$$

hold. By virtue of the assumption $f \in L^2_\mu$, condition (4) holds for almost every $t \in (-\pi, \pi)$. Namely, if we write μ' for a fixed Radon–Nikodym derivative of (the absolutely continuous part of) μ, and if we denote by E_w the set of t's where

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} |f(\theta) - w|^2 \, d\mu(\theta) \neq |f(t) - w|^2 \, \mu'(\theta),$$

and by E the set of t's where

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} d\mu(\theta) \neq \mu'(\theta),$$

then (4) holds for every $t \in (-\pi, \pi)$ not belonging to E and to any of the sets E_w, w a complex rational. This argument is carried out for the case $d\mu(t) = dt$ in more detail in [10, Vol. I, Theorem II.11.3, p. 65] on account of a discussion of Lebesgue points.

As for condition (5), this holds for almost every $t \in (-\pi, \pi)$ provided μ is in the Szegő class; that is, when

$$\int_{-\pi}^{\pi} \log \mu'(\theta) \, d\theta > -\infty, \tag{6}$$

according to [6, Theorem 1 and formula (5), p. 147] (a more precise result is found in [7, Theorems 1 and 7, pp. 435 and 449]). It is interesting to note that (6) is exactly the condition for the set of polynomials of $e^{i\theta}$ not to be complete in L^2_μ (cf. [3, Theorem V.2.3, p. 200, and Theorem V.2.5, p. 204]). Formula (6) is a consequence of (2); in fact, (6) is equivalent to the condition

$$\sum_{\nu=0}^{\infty} |\phi_\nu(0)|^2 < \infty;$$

see, e.g., [3, Formulas (V.2.6) and (V.2.7), p. 200, and Theorem V.2.5, p. 204] or [8, Formula (11.3.6) and Theorem 11.3.3, pp. 290–291].

While we will not use this fact below, condition (2) implies that even more than (5) is true; namely, it implies that

$$\lim_{n\to\infty} e^{-int}\phi_n(e^{it}) \qquad \text{exists and is finite} \tag{7}$$

for almost every $t\in[-\pi,\pi)$ with respect to the measure μ, hence, a fortiori (since $\mu'>0$ almost everywhere in view of (6)), for almost every $t\in[-\pi,\pi)$ with respect to the Lebesgue measure. This was pointed out to me by Paul Nevai, and his reasoning is as follows. It can easily be shown using summation by parts that (2) implies that

$$\sum_{\nu=1}^{\infty} |\phi_\nu(0)|^2 \log^2 \nu < \infty;$$

hence

$$\sum_{\nu=0}^{\infty} \overline{\phi_\nu(0)}\,\phi_\nu(e^{it})$$

converges for almost every t with respect to μ by the Menšov–Rademacher theorem (see, e.g., [10, Vol. II, Theorem XIII.10.21, p. 193] or [1, Theorem 2.3.2, p. 80]).[1] As for the partial sums of this series, we have

$$\sum_{\nu=0}^{n} \overline{\phi_\nu(0)}\,\phi_\nu(z) = \kappa_n\phi_n^*(z),$$

where

$$\phi_n^*(z) = z^n\overline{\phi_n(1/\bar{z})}; \tag{8}$$

see, e.g., [3, Theorem V.1.8, p. 195] or [8, Theorem 11.3.2, p. 290]. In view of the same theorems in [3] or [8], we have

$$\kappa_n^2 = \sum_{j=0}^{n} |\phi_j(0)|^2, \tag{9}$$

[1] Alexits [1] describes the Menšov–Rademacher Theorem for real-valued orthogonal systems, but the proof he gives works for complex-valued systems equally well; most other sources also deal only with real-valued systems. Zygmund [10] considers complex-valued orthogonal systems, but only with respect to the Lebesgue measure; this is not a genuine limitation, however, since the case of an arbitrary finite positive Borel measure can be obtained via a change of variables.

and so κ_n tends to a finite positive limit as $n \to \infty$, according to (2). Hence $\phi_n^*(e^{it})$ converges to a finite limit for almost every t with respect to μ. Thus (7) holds in view of (8). In particular, we have

$$\sup_n |\phi_n(e^{it})| < \infty \tag{10}$$

for almost every $t \in [-\pi, \pi)$.

2. Proof of the Main Results

Throughout this section we write $\zeta = e^{i\theta}$ and $z = e^{it}$. θ will usually vary, and $t \in (-\pi, \pi)$ will be a fixed quantity for which (4) and (5) are satisfied. We write $\mathscr{P}$ for the closure in L^2_μ of the set of polynomials of $e^{i\theta}$. After the proof of Theorem 1, we make some comparisons with Freud's proof of [2, Satz I, p. 84]. The following estimate is crucial in the proof:

Lemma 2. *Assume $f \in L^2_\mu$ and* (4) *is satisfied. Let $r_n = 1 + 1/n$. Then*

$$\int_{-\pi}^{\pi} |f(\theta) - f(t)|^2 \left| \frac{1}{\zeta - r_n z} \right|^2 d\mu(\theta) = o(n) \tag{11}$$

as $n \to \infty$.

Proof. For the sake of simplicity we assume $t = 0$ in the proof. We break up the interval of integration into the parts $S_1 = [-1/n, 1/n]$ and $S_2 = [-\pi, \pi) \backslash S_1$. The integral on S_1 is less than

$$n^2 \int_{S_1} |f(\theta) - f(0)|^2 \mu(\theta),$$

and this is $o(n)$ in view of (4) with $t = 0$. As for the integral on S_2, writing $F = F_0$ for the function defined in (4), we obtain by integrating by parts that

$$\begin{aligned}
&\frac{4}{\pi^2} \int_{S_2} |f(\theta) - f(0)|^2 \left| \frac{1}{e^{i\theta} - r_n} \right|^2 d\mu(\theta) \\
&\leqslant \int_{S_2} |f(\theta) - f(0)|^2 \frac{1}{\theta^2} d\mu(\theta) \\
&= \frac{F(\pi)}{\pi^2} - F\left(\frac{1}{n}\right) n^2 + F\left(\frac{-1}{n}\right) n^2 - \frac{F(-\pi)}{\pi^2} + \int_{S_2} \frac{F(\theta)}{\theta^3} d\theta.
\end{aligned}$$

The right-hand side here is $o(n)$ according to (4) with $t = 0$. ∎

Before we turn to the proof of Theorem 1, we state a useful consequence of the orthogonality relations (1): for all integers m and n with $0 \leqslant m \leqslant n$ we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \zeta^m \overline{\phi_n(\zeta)}\, d\mu(\theta) = \frac{\delta_{mn}}{\kappa_n}. \tag{12}$$

This follows from (1) simply by representing ζ^m as a linear combination of $\phi_k(\zeta)$ for $k \leqslant m$.

Proof of Theorem 1. We have

$$\begin{aligned} E_\nu &\stackrel{\text{def}}{=} s_\nu(t) - f(t) = -f(t) + \sum_{j=0}^{\nu} c_j \phi_j(z) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\theta) - f(t))\, K_{\nu+1}(\zeta, z)\, d\mu(\theta), \end{aligned} \tag{13}$$

where c_j are the expansion coefficients of f, and

$$K_{\nu+1}(\zeta, z) = \sum_{j=0}^{\nu} \overline{\phi_j(\zeta)}\, \phi_j(z) = \frac{\overline{\phi_{\nu+1}^*(\zeta)}\, \phi_{\nu+1}^*(z) - \overline{\phi_{\nu+1}(\zeta)}\, \phi_{\nu+1}(z)}{1 - \bar{\zeta} z}; \tag{14}$$

here the second equality is Szegő's modified Christoffel–Darboux formula (see [8, Formula (11.4.5), p. 293] or [3, Lemma V.1.10, p. 196]), and the starred polynomials are defined by (8) above.

In estimating E_ν, we move the singularity on the right-hand side of (14) outside the unit circle,

$$E_\nu = E_{1\nu} + E_{2\nu}, \tag{15}$$

where, writing $r = 1 + 1/n$ for a given n,

$$E_{1\nu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\theta) - f(t))\, \frac{-z}{n(\zeta - rz)}\, K_{\nu+1}(\zeta, z)\, d\mu(\theta) \tag{16}$$

and

$$E_{2\nu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(\theta) - f(t))\, \frac{\zeta - z}{\zeta - rz}\, K_{\nu+1}(\zeta, z)\, d\mu(\theta). \tag{17}$$

$E_{1\nu}$ can easily be estimated by Schwarz's inequality,

$$|E_{1\nu}| \leqslant \left(\frac{1}{2\pi n^2}\int_{-\pi}^{\pi} |f(\theta)-f(t)|^2 \left|\frac{z}{\zeta - rz}\right|^2 d\mu(\theta)\right)^{1/2}$$
$$\times\left(\int_{-\pi}^{\pi} |K_{\nu+1}(\zeta, t)|^2 \, d\mu(\theta)\right)^{1/2}$$
$$= o(n^{-1/2})\left(\sum_{j=0}^{\nu} |\phi_j(t)|^2\right)^{1/2} = o(1) \tag{18}$$

as $n \to \infty$. Here the first equality holds by Lemma 2, and the second one, by (5); furthermore, the second integral after $\leqslant$ symbol was evaluated by using the orthogonality properties (1) of ϕ_j.

In estimating $E_{2\nu}$, note that by the second equality in (14) we have

$$E_{2\nu} = \phi^*_{\nu+1}(z)\, A_\nu - \phi_{\nu+1}(z)\, B_\nu, \tag{19}$$

where

$$A_\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(\theta)-f(t)) \frac{\zeta}{\zeta - rz} \overline{\phi^*_{\nu+1}(\zeta)}\, d\mu(\theta), \tag{20}$$

and

$$B_\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(\theta)-f(t)) \frac{\zeta}{\zeta - rz} \overline{\phi_{\nu+1}(\zeta)}\, d\mu(\theta). \tag{21}$$

We estimate sums involving A_ν and B_ν. The latter is easier. Namely, by Bessel's inequality,

$$\sum_{\nu=0}^{n-1} |B_\nu|^2 \leqslant \int_{-\pi}^{\pi} \left|(f(\theta)-f(t)) \frac{\zeta}{\zeta - rz}\right|^2 d\mu(\theta) = o(n) \tag{22}$$

as $n \to \infty$; the equality here holds according to Lemma 2 (recall that r was chosen the same as the r_n of this lemma; the B_ν's depend on n, even though this dependence is not indicated explicitly). Using Schwarz's inequality, (5) now gives

$$\sum_{\nu=0}^{n} |\phi_{\nu+1}(z)|\, |B_\nu| \leqslant \left(\sum_{\nu=0}^{n} |\phi_{\nu+1}(z)|^2\right)^{1/2} \left(\sum_{\nu=0}^{n} |B_\nu|^2\right)^{1/2}$$
$$= O(n^{1/2})\, o(n^{1/2}) = o(n). \tag{23}$$

The estimation of the analogous sum involving the A_ν's is more complicated, since the polynomials ϕ^*_ν are not orthogonal. Since $f(\theta) -$

$f(t) \in \mathscr{P}$ by assumption ($\mathscr{P}$ was defined at the beginning of this section), and $1/(\zeta - rz)$ is a uniformly convergent limit of polynomials of ζ, we also have

$$(f(\theta) - f(t)) \frac{1}{\zeta - rz} \in \mathscr{P}. \tag{24}$$

With the orthogonal expansion

$$(f(\theta) - f(0)) \frac{1}{\zeta - rz} \sim \sum_{j=0}^{\infty} a_j \phi_j(\zeta)$$

we have

$$\begin{aligned} A_\nu &= \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\phi^*_{\nu+1}(\zeta)}\, \zeta \sum_{j=0}^{N} a_j \phi_j(\zeta)\, d\mu(\theta) \\ &= \lim_{N \to \infty} \sum_{j=0}^{N} a_j \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\phi^*_{\nu+1}(\zeta)}\, \zeta \phi_j(\zeta)\, d\mu(\theta). \end{aligned} \tag{25}$$

It now follows from the equation

$$\int_{-\pi}^{\pi} \overline{\phi^*_k(\zeta)}\, \zeta p(\zeta)\, d\mu(\theta) = 0,$$

true for every polynomial p of degree $< k$ (this relation is a simple consequence of (8) and the complex conjugate of (12); see, e.g., [3, Lemma V.1.9, p. 196]), that the integrals on the right-hand side are zero unless $j > \nu$. For $j > \nu$ the integrals can be evaluated as follows:

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\phi^*_{\nu+1}(\zeta)}\, \zeta \phi_j(\zeta)\, d\mu(\theta) \\ &\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\phi^*_{\nu+1}(0)}\, \zeta \phi_j(\zeta)\, d\mu(\theta) \\ &\quad = \frac{\kappa_{\nu+1}}{2\pi\kappa_j} \left(\int_{-\pi}^{\pi} \kappa_{j+1} \phi_{j+1}(\zeta)\, d\mu(\theta) - \int_{-\pi}^{\pi} \phi_{j+1}(0)\, \phi^*_{j+1}(\zeta)\, d\mu(\theta) \right). \end{aligned}$$

To get the first equality we used the fact that the that only the constant term of $\phi^*_{\nu+1}$ gives a nonzero contribution to the integral in view of the complex conjugate of (12). For the second equality, we used the equation $\phi^*_{\nu+1}(0) = \kappa_{\nu+1} (= \bar{\kappa}_{\nu+1})$ and the recurrence relation

$$\kappa_j \zeta \phi_j(\zeta) = \kappa_{j+1} \phi_{j+1}(\zeta) - \phi_{j+1}(0)\, \phi^*_{j+1}(\zeta);$$

for the latter, see, e.g., [8, Formula (11.4.6), p. 293]. The first integral on the right-hand side of the above expression is 0 in view of the orthogonality relations (1); in evaluating the second integral, we use (8), the fact that $1/\bar{\zeta}=\zeta$ as $|\zeta|=1$, and (12). Thus the right-hand side equals

$$\frac{\kappa_{\nu+1}}{2\pi\kappa_j}\phi_{j+1}(0)\int_{-\pi}^{\pi}\zeta^{j+1}\overline{\phi_{j+1}(\zeta)}\,d\mu(\theta)=\frac{\kappa_{\nu+1}}{\kappa_j\kappa_{j+1}}\phi_{j+1}(0).$$

Hence

$$A_\nu=\sum_{j=\nu+1}^{\infty}\frac{\kappa_{\nu+1}}{\kappa_j\kappa_{j+1}}\phi_{j+1}(0)a_j.$$

Here $\kappa_{\nu+1}/(\kappa_j\kappa_{j+1})$ is bounded independently of ν and j in view of (2) and (9). Hence, for some positive constants C and C', we have[2]

$$\begin{aligned}\sum_{\nu=0}^{n-1}|\phi^*_{\nu+1}(z)|\,|A_\nu| &\leqslant C\sum_{\nu=0}^{n-1}\sum_{j=\nu+1}^{\infty}|\phi_{j+1}(0)|\,|a_j|\,|\phi_{\nu+1}(z)|\\ &=C\sum_{j=1}^{\infty}|\phi_{j+1}(0)|\,|a_j|\sum_{\nu=0}^{\min(j-1,n-1)}|\phi_{\nu+1}(z)|\\ &\leqslant C\sum_{j=1}^{\infty}|\phi_{j+1}(0)|\,|a_j|\,(\min(j,n))^{1/2}\left(\sum_{\nu=0}^{\min(j-1,n-1)}|\phi_{\nu+1}(z)|^2\right)^{1/2}\\ &\leqslant C'\sum_{j=1}^{n}j\,|\phi_{j+1}(0)|\,|a_j|+C'n\sum_{j=n+1}^{\infty}|\phi_{j+1}(0)|\,|a_j|;\end{aligned}\tag{26}$$

the second inequality here follows by Schwarz's inequality, and the third one holds in view of (5).

The second sum on the right-hand side is easy to estimate by Schwarz's inequality:

$$\begin{aligned}\sum_{j=n+1}^{\infty}|\phi_{j+1}(0)|\,|a_j| &\leqslant\left(\sum_{j=n+1}^{\infty}|\phi_{j+1}(0)|^2\right)^{1/2}\left(\sum_{j=n+1}^{\infty}|a_j|^2\right)^{1/2}\\ &=O(1/\sqrt{n})\,o(\sqrt{n})=o(1).\end{aligned}\tag{27}$$

The first equality here follows by (2) and the relation

$$\sum_{j=1}^{\infty}|a_j|^2=o(n).\tag{28}$$

The latter is a consequence of Bessel's inequality and Lemma 2.

[2] This is practically the only point in the proof where there would be a slight simplification if we were to use (10) instead of (5). The gain in the following calculation would be minor. The loss, owing to the fact that (5) might hold at more points than (10), would also be minor.

In order to estimate the first sum on the right-hand side of (26), write

$$\eta_j = \sum_{k=j+1}^{\infty} |\phi_k(0)|^2.$$

We can estimate the sum in question by using Schwarz's inequality and then summation by parts:

$$\begin{aligned}
\sum_{j=1}^{n} j\,|\phi_{j+1}(0)|\,|a_j| &= \sum_{j=1}^{n} j\sqrt{\eta_j - \eta_{j+1}}\,|a_j| \\
&\leqslant \left(\sum_{j=1}^{n} j^2(\eta_j - \eta_{j+1})\right)^{1/2} \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \\
&= \left(-n^2\eta_{n+1} + \sum_{j=1}^{n} (j^2 - (j-1)^2)\eta_j\right)^{1/2} \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2} \\
&\leqslant \left(\sum_{j=1}^{n} O(1)\right)^{1/2} (o(n))^{1/2} = o(n).
\end{aligned}$$

The second inequality here follows by (2) and (28). Putting this and (27) together with (26), we obtain

$$\sum_{\nu=0}^{n} |\phi^*_{\nu+1}(z)|\,|A_\nu| = o(n). \tag{29}$$

Using this and (23), by (19) we obtain

$$\sum_{\nu=0}^{n} |E_{2\nu}| = o(n).$$

Now (3) follows from this relation and (18) (cf. (13)). The proof of the theorem is complete. ▌

Comparisons with Freud's Proof. Freud obtained a stronger result in [2, Satz I, p. 84] for orthogonal polynomials on the interval $[-1, 1]$ than our Theorem 1 for the unit circle. His proof, which followed closely that of Tandori [9], avoided some technical complications in view of the different form of the Christoffel–Darboux formula for the real line. In his proof, he estimated the integral analogous to the right-hand side of our formula (13) by cutting the interval of integration into two parts as we did it in the proof of Lemma 2; the proof of this lemma follows Freud's reasoning in [3, p. 86]. The crucial difference here is that formula (20) involves the polynomials $\phi^*_{\nu+1}$, which are not orthogonal, so the estimation of sums involving the A_ν's is more difficult and less precise. An important relation for this estimation is (24), which is needed for the validity of (25). This is why, instead of cutting the interval of integration in (13), we moved the singularity, as was done in (16) and (17). Interesting observations on the background of Freud's and Tandori's proofs can be found in [1, Sect. 3.4, p. 212].

Acknowledgments

The author expresses his gratitude to Professor Paul Nevai of The Ohio State University, Columbus, Ohio, and Professor Vilmos Totik of the University of South Florida, Tampa, Florida, for their comments.

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[3] The authors of this and the preceding item are the same person; the different renderings of his name follow the way the name is given in the quoted items.